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THE CONFORMAL INVARIANCE OF HUYGEN'S PRINCIPLE

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Introduction

In this paper we study certain hyperbolic equations satisfying the strong Huygen's principle (SHP) in the sense that the elementary solutions are supported on a hypersurface. Recently Lax and Phillips [7] have discussed the wave equation on odd-dimensional rank-one symmetric spaces and shown how certain coordinate transformations yield the SHP by reduction to the wave equation in Euclidean space. Also Helgason [3] has given examples of compact groups and symmetric spaces on which the SHP holds for the natural wave equations.

We will adopt the approach begun in [12] and indirectly suggested in [7] to show that, if there is a conformal transformation between two Lorentz manifolds M_1 and M_2 of constant scalar curvature, then the wave equation in M_1 satisfies the SHP if and only if the same is true in M_2. In this case the transformation maps characteristic cones to characteristic cones, so it is perhaps not surprising that it also provides a transformation between the elementary solutions. As a corollary we get new examples of Lorentz manifolds on which the wave equation satisfies the SHP.

More generally we derive similar principles for certain ultrahyperbolic equations on pseudo-Riemannian manifolds of constant scalar curvature, and we discuss the SHP for the Dirac and Maxwell equations. Finally we point out an elementary connection between causality-preserving transformations and automorphisms of complex domains.

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1. Let M be a pseudo-Riemannian manifold of dimension n with pseudo-metric g and constant scalar curvature K. Consider the Laplace-Beltrami operator $\square$ on M, [12], [13], and the generalized wave operator

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$$L = \square + \frac{n-2}{4(n-1)} K. \tag{1}$$

As shown in [12] this operator is conformally quasi-invariant in the following sense: if T: $V \to W$ is a conformal diffeomorphism between two open sets in M, so that the pull-back metric $T^*g = \gamma g$ for some positive function γ (the Jacobian of T raised to the $(2/n)$th power), then

$$L\gamma^{(n-2)/4}T^* = \gamma^{(n+2)/4}T^*L.$$

In other words, for every smooth function f on W we have

$$L\big(\gamma(x)^{(n-2)/4}f(T(x))\big) = \gamma(x)^{(n+2)/4}(Lf)(T(x)), \tag{2}$$

which in particular says that the null-space of L is invariant under the action

$$f(x) \to \gamma(x)^{(n-2)/4}f(T(x)). \tag{3}$$

The resulting representation of the conformal group of M has in special cases been studied in detail [4], [12], [11]. Note that (3) fails to be unitary in $L^2(M)$.

Now suppose we are given two such manifolds M_1 and M_2 of the same dimension n with data as above, i.e., pseudo-metrics g_1 and g_2, constant scalar curvatures K_1 and K_2 and "wave" operators L_1 and L_2 as in (1). We wish to generalize the covariance in (2) to mappings between M_1 and M_2: Let V_1 and V_2 be open sets in M_1 and M_2, and T: $V_1 \to V_2$ a conformal diffeomorphism, $T^*g_2 = \gamma g_1$. Then we have that L_1 and L_2 are intertwined via T as follows:

Proposition 1. *Under the hypotheses above we have for any smooth function f on V_2 that*

$$L_1\big(\gamma(x)^{(n-2)/4}f(T(x))\big) = \gamma(x)^{(n+2)/4}(L_2f)(T(x)). \tag{4}$$

In particular, solutions to $L_2 f = 0$ on V_2 are mapped to solutions of $L_1\tilde{f} = 0$ on V_1 via

$$\tilde{f}(x) = \gamma(x)^{(n-2)/4}f(T(x)). \tag{5}$$

Proof. $\gamma(x)$ being related to the Jacobian of a conformal transformation between spaces of constant scalar curvature has to satisfy the following (nonlinear) differential equation (in [5] this is carried out for positive definite metric on a compact manifold, and the general case follows by similar differential geometric arguments)

$$\square_1 h = -\frac{n-2}{4(n-1)}\big(K_1 h - K_2 h^{(n+2)/(n-2)}\big), \tag{6}$$

where $h(x) = \gamma(x)^{(n-2)/4}$. On the other hand, by Lemma 3.1 in [12] we have

$$\square_1 hT^*f - \gamma hT^*\square_2 f = (\square_1 h)T^*f, \tag{7}$$

so that by combining (6) and (7) we get

$$\left(\square_1 + \frac{n-2}{4(n-1)} K_1\right) hT^*f = \gamma hT^*\left(\square_2 + \frac{n-2}{4(n-1)} K_2\right) f,$$

which is exactly (4). Note that one can prove that (6) is also a sufficient condition for a positive function $h(x)$ to be equal to $\gamma(x)^{(n-2)/4}$ corresponding to some conformal transformation of M_1 into a space of scalar curvature K_2.

Remark 2. (a) This proposition was established in [12] in many special cases and has been a natural conjecture.

(b) Equation (6) is actually invariant under conformal transformations in M_1 of the form (3) by virtue of the quasi-invariance (2).

It is clear that (5) also sets up a correspondence between distributions in V_1 and V_2 and that we still have (4) and a correspondence between solutions to $L_2 f = 0$ and $L_1 h = 0$. (5) preserves singular support, so if f is supported along a hypersurface, so is h. This is the well-known use of coordinate transformations as in (5) to arrive at a SHP in one manifold by knowing it in another. The virtue of conformal transformations is that they preserve characteristic cones (in the case of Lorentz manifolds, i.e., metrics of signature $(+-\cdots-)$) for the Laplace-Beltrami operators, so that the local behavior of solutions is the same in conformally equivalent regions. Hence in any manifold locally conformally equivalent to the examples given in [3] (e.g., $\mathbf{R} \times SU(2n)$) we have the SHP.

Now let M_0 be $\mathbf{R}^n$ (n even $\geqslant 4$) with its standard Lorentz metric

$$ds^2 = dx_1^2 - dx_2^2 - \cdots - dx_n^2$$

and wave operator $L_0 = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \cdots - \partial^2/\partial x_n^2$. This has retarded and advanced fundamental solutions supported on the boundary of the forward (resp. backward) light cone. As a first example of our techniques, let us construct a conformal transformation of M_0 into the de Sitter space

$$H = \left\{(y_0, y_1, \cdots, y_n) \in \mathbf{R}^{n+1} | y_0^2 + y_1^2 - y_2^2 - \cdots - y_n^2 = 1\right\},$$

which is a homogeneous space for $O(2, n-1)$ with isotropy group $O(1, n-1)$ (the Lorentz group). Let

$$T(x) = \frac{\left(1 - \frac{1}{4}x^2, x\right)}{1 + \frac{1}{4}x^2},$$

where $x^2 = x_1^2 - x_2^2 - \cdots - x_n^2$. Then [11] T is a conformal diffeomorphism from $\{x | 1 + \frac{1}{4}x^2 \neq 0\}$ onto $\{y | y_0 \neq -1\}$ (in particular from a neighborhood of the origin of M_0). On H we have $L_H = \Box_H - ((n-2)/2)^2$ and $\gamma(x) = (1 + \frac{1}{4}x^2)^{-2}$ so that

$$L_0 \gamma^{(n-2)/4} T^* = \gamma^{(n+2)/4} T^* L_H. \tag{8}$$

Since a δ-signal in M_0 propagates along the boundary of the light-cone, i.e., along light-rays, also in H sharp signals will remain confined to light-rays. This is true locally by (8) and hence eveywhere by the homogeneity of H. In particular spacelike Cauchy data of small support will propagate to form lacunae where the solution is zero. Also in $\tilde{H}$, the universal covering of H, homogeneous for the universal covering group of $O(2, n-1)$, the SHP is satisfied.

More generally, suppose M_1 and M_2 are globally hyperbolic [1] Lorentzian manifolds of constant scalar curvature (e.g., $\tilde{H}$ and M_0 above) so that each has defined on it unique global retarded and advanced fundamental solutions. For example $E_1^+(x, x')$ is the distribution on $M_1 \times M_1$ satisfying

$$L_1 E_1^+(x, x') = \delta(x, x')$$

(acting on second coordinate) where $\delta(x, \cdot)$ is the Dirac δ-distribution at the point x and for a fixed x, $E_1^+(x, x')$ has forward time-like support. Similarly for E_1^-, E_2^+, E_2^-. Then [12]

Proposition 3. *Let T: $M_1 \to M_2$ be conformal with M_1 and M_2 as above. Then*

$$\gamma(x)^{(n-2)/4} E_2^+(T(x), T(x'))\gamma(x)^{(n-2)/4} = E_1^+(x, x'), \tag{9}$$

(and the same for E^-).

Considering the conformal compactification of M_0 ($n = 4$) we get as a corollary the advanced fundamental solution on $\mathbf{R} \times S^3$ to be (at $\tau = \rho = 0$)

$$E^+(\tau, \rho) = \frac{1}{4\pi} \frac{\delta(\tau - \rho),}{\sin \rho},$$

where ρ is the polar angle on S^3 from the north pole.

In the setting of Proposition 3 the Greens function

$$G_1(x, x') = E_1^+(x, x') - E_1^-(x, x')$$

will satisfy a relation similar to (9). Define for a test function $\varphi(x)$ on M_1 (and similarly in M_2)

$$(\varphi, \varphi)_1 = \int_{M_1}\int_{M_1} G_1(x, x')\varphi(x')\overline{\varphi(x)}\, dx'dx. \tag{10}$$

Then by virtue of the covariance of G we have for any test function ψ on M_2 and $\varphi = \gamma^{(n+2)/4} T^* \psi$

$$\begin{aligned}(\varphi, \varphi)_1 &= \int_{M_1}\int_{M_2} G_1(x, x')\gamma(x')^{(n+2)/4}\psi(T(x'))\,\overline{\psi(T(x))}\,\gamma(x)^{(n+2)/4}\,dx'\,dx \\ &= \int_{M_1}\int_{M_2} \gamma(x)^{(n-2)/4}G_2(T(x), y')\psi(y')\,\overline{\psi(T(x))}\,\gamma(x)^{(n+2)/4}\,dy'\,dx \\ &= \int_{M_2}\int_{M_2} G_2(y, y')\psi(y')\,\overline{\psi(y)}\,dy'\,dy \\ &= (\psi, \psi)_2.\end{aligned}$$

Hence the Hermitian form (10) is conformally invariant, and if positive (as on M_0, see also [8]) it defines an invariant unitary structure on the space of test functions. Actually this invariant form is defined on the space of solutions f to the wave equation via

$$f(x) = \int_M G(x, x')\varphi(x')\,dx',$$

where the action of the conformal group of f is the one given by (3). This action then leaves invariant $(f, f) = (\varphi, \varphi)$. In some cases [11], [4] there results an irreducible unitary continuous representation of the conformal group of the manifold on the space of solutions to $Lf = 0$.

Finally let us give a list of (conformally flat) locally symmetric Lorentzian manifolds of constant scalar curvature which have open dense subsets which are conformal images of open sets in the linear space M_0. In particular they all satisfy the SHP for the wave operator [12]. The cases are (n even $\geqslant 4$):

(a) $\mathbf{R} \times S^{n-1}$,
(b) $O(2, n-1)/O(1, n-1)$,
(c) $O(1, n)/O(1, n-1)$,
(d) $O(1, q)/O(q) \times O(1, n-q)/O(1, n-q-1)$ $(q = 0, 1, \cdots, n-1)$,
(e) $O(2, q)/O(1, q) \times O(n-q)/O(n-q-1)$, $(q = 1, 2, \cdots, n-2)$,

and any covering of these.

If in (d) $q = n - 1$, then the manifold is

$$O(1, n-1)/O(n-1) \times O(1, 1)/O(1),$$

where the first part is the "space" part, and the second the "time" part. By an extra coordinate transformation (nonconformal) the "time" part can be made S^1 or its covering $\mathbf{R}$ so that we also get the SHP on $\mathbf{R} \times O(1, n-1)/O(n-1)$ as in [3], [7].

Remark 4. Suppose $\mathbf{R}$ is equipped with the metric $h(s)^2 ds^2$ where h is positive; the corresponding Laplace-operator is $h^{-1}(\partial/\partial s)h^{-1}(\partial/\partial s)$. On the other hand if $a\colon \mathbf{R} \to \mathbf{R}$ is monotone and bijective, we have $\partial^2/\partial s^2 f(a(s)) = a'(s)^2 f''(a(s)) + a''(s)f'(a(s))$. Hence the change of coordinate via a maps

$\partial^2/\partial s^2$ into $h^{-1}(\partial/\partial s)h^{-1}(\partial/\partial s)$ if and only if $a'(s)^2 = h(s)^{-2}$ and $a''(s) = h(s)^{-1}(h^{-1})'(s) = -h'(s)h(s)^{-3}$. But the second equation is implied by the first which can simply be solved by $a(s) = \int_0^s h(t)^{-1}dt$.

As another special case of the above we mention $\mathbf{R} \times \widetilde{SL}(2, \mathbf{R})$ (universal covering) where minus the Killing form on $sl(2, \mathbf{R})$ induces a pseudometric of signature $(+--)$ which together with the (negative) metric $-ds^2$ on $\mathbf{R}$ makes this space into a Lorentz manifold on which $L = C - \partial/\partial s^2 + 1$ satisfies the SHP. Here C is the Casimir operator of $SL(2, \mathbf{R})$. In fact, this case arises in (e) when $n = 4$ and $q = 2$.

Remark 5. The spectrum of L on $(SL(2, \mathbf{R}) \times S^1)/\mathbf{Z}_2 \simeq U(1, 1)$ plays a role in the decomposition of holomorphic discrete series representations of $SU(2, 2)$ when restricted to $S(U(1, 1) \times U(1, 1))$, quite analogous to the case of restrictions to the maximal compact subgroup $S(U(2) \times U(2))$.

2. Both the Dirac equation (for zero mass) and Maxwell's equations (in vacuum) are known to satisfy the SHP in n-dimensional Minkowski space (n even $\geqslant 4$). This is readily seen from the fact that the components of the fields satisfy the wave equation.

The conformal invariance of Dirac's equation has been studied in [6]. Let us summarize what we shall need: Assume M is a Lorentz manifold of dimension $2m$ with $H^2(M, \mathbf{Z}_2) = 0$ (so that the spin bundle is well-defined). Denote by S the spin bundle (with complex fiber dimension 2^m) with structure group $\mathrm{Spin}(1, 2m-1)$, a double covering of $SO(1, 2m-1)$. The Dirac operator D acts on sections of S by composing the covariant differentiation in S with Clifford multiplication, in local coordinates $D = \sum_{i=1}^{2m} \gamma_i \nabla_{e_i}$ where $\{e_i\}$ is an orthonormal Lorentz frame at $x \in M$ and $\gamma_i = \gamma(e_i)$, the γ-matrices corresponding to $\gamma\colon T_xM \to S_x \otimes S_x^*$.

Suppose $T\colon M \to M$ is conformal (maybe only in an open set), then the differential $T_*\colon T_xM \to T_{T(x)}M$ preserves the inner product up to a scalar, in particular it has an action on the spin fiber (via the spin covering $\mathrm{Spin}(1, 2m-1) \to SO(1, 2m-1)$)

$$\tau(T_*)\colon S_x \to S_{T(x)}.$$

Now the action on a section ψ of S is going to be

$$V(T)\colon \psi(x) \to \gamma_{T^{-1}}(x)^{(2m-1)/4}\tau(T_*)\psi(T^{-1}(x)).$$

We then have (the integrated form of the corresponding infinitesimal formula in [6], see also [4] for the flat case)

$$DV(T) = (\gamma_{T^{-1}})^{1/2} V(T) D. \tag{11}$$

(11) also holds when T maps from one manifold into another. Clearly then,

$$D\psi = 0 \tag{12}$$

is conformally invariant, and the SHP holds for the hyperbolic equation (12) if and only if it holds on a conformally equivalent manifold. In particular the SHP holds for (12) on, e.g., $\mathbf{R} \times S^3$.

Remark 6. If M is a reductive coset space G/H, we can parallelize S as follows [9]: Let $\mathbf{p}$ be the complement of the Lie algebra of H and assume the actions

$$\alpha\colon H \to SO(\mathbf{p}),$$
$$\tilde{\alpha}\colon H \to \mathrm{Spin}(\mathbf{p}) \to \mathrm{Aut}\ L,$$

where L is the fiber of the spin bundle

$$S = G \times_H L.$$

Let $\Gamma(S)$ be the smooth sections and

$$\eta\colon \Gamma(S) \to L \otimes C^\infty(G),$$

where η is (well) defined by

$$\{g, \eta(\psi)(g)\} = \psi(gH)$$

for $\psi \in \Gamma(S)$. $\eta(\psi)$ satisfies for $h \in H$ and $g \in G$

$$h \cdot \eta(\psi)(gh) = \eta(\psi)(g).$$

Then the action of D is

$$\eta(D\psi) = \sum_{i=1}^{2m} (\gamma(X_i) \otimes \nu(X_i))\eta(\psi),$$

where $\{X_i\}$ is a Lorentz frame at the origin, and ν is the usual identification of $\mathbf{p}$ with vector fields on G/H. As proven in [9] in the positive definite case we also here have

$$\eta(D^2\psi) = -(I \times \nu(C) + c)\eta(\psi),$$

where C is the Casimir operator (so that $\nu(C)$ is the Laplace-Beltrami operator on G/H), and c is the constant from before. Hence the square of the Dirac operator in this way becomes the wave operator. We conclude that if the wave equation in G/H satisfies the SHP, so does Dirac's equation (this is the case [3] e.g. on $\mathbf{R} \times K$, K a compact semi-simple Lie group of odd dimension).

For Maxwell's equations on forms ω of degree m, $\dim M = 2m$,

$$d\omega = 0, \quad \delta\omega = 0,$$

where $\delta = *d*$ is the adjoint to $d\colon \Lambda^k M \to \Lambda^{k+1} M$, we have for T conformal, $dT^* = T^*d$ (always) and $\delta T^* = \gamma T^* \delta$ (when acting on forms of degree m).

Hence $\omega \to T^*\omega$ preserves solutions to Maxwell's equations, even when going from one manifold to another. We conclude that Maxwell's equations satisfy the SHP on all manifolds locally conformally equivalent to even-dimensional Minkowski space.

3. The role of conformal transformations as seen above is to preserve the characteristic cones, in other words, they are causality preserving. A manifold M endowed with a smooth field of proper closed convex cones (one cone in each tangent space) is said to be causal [10] with the natural induced time-like and space-like ordering. The group preserving the causal structure is called the causal group of M, and in the cases above it essentially coincides with the conformal group. The cone-field however need not be derived from a pseudo-metric (in which case the cone is of rank 2) but can be of higher rank, as e.g. in $H(n)$, all Hermitian $n \times n$ matrices, where the cone of all positive definite matrices has rank n. Another example is $U(n)$ with the left-invariant cone field induced from $H(n)$, the Lie algebra (up to multiplication by i). There is in some cases a close connection between causal manifolds and their causal groups on the one hand, and open subdomains of $\mathbf{C}^n$ and their automorphism groups on the other.

Suppose D is a Hermitian symmetric space of tube type, and M its Shilov boundary, $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} D = n$. At a point $x \in M$ the tangent space T_xM has a complement K_x so that

$$T_xM + K_x = \mathbf{C}^n, \tag{12}$$

and (12) is the local splitting in purely real and imaginary coordinates. The direction from x into D is given by a cone V in K_x (the cone defining the tube domain).

Remark 7. Realizing D as a tube domain $\{x + iy | y \in V\}$ (e.g., the upper half-plane) the splitting (12) is simply in x and y, the real and imaginary coordinates.

Suppose T: $D \to D$ is biholomorphic, and consider $x \in M$. M is also stable under T and the differential on M

$$T_*: T_xM \to T_{T(x)}M.$$

T_* on D is complex-linear and preserves D so that T_* is of the form (relative to the splitting (12) at x and $T(x)$)

$$T_* = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A = B$ by complex linearity, and A is the differential of T on M. Here B must preserve V (since T preserves D), hence A preserves V which proves that T on M is locally causality preserving, where we equip M with the cone-field defined by V. In other words, locally we have that i times a timelike direction on M is a direction into D. Note also that in the same way one gets that a holomorphic transformation from a domain of tube type to another, on the Shilov boundary is causality-preserving (as for example the Cayley transform).

It would seem that reversing the argument (given a causality-preserving transformation on M, extend it to an isomorphism of D) requires a little more work.

The interpretation of Huygen's principle in terms of objects on D is apparently not known; but it seems that hyperbolicity and causality on the boundary somehow reflects the geometry of D.

References

[1] Y. Choquet-Bruhat, *Batelle Rencontres* 1967 (Editors C. de Witt and J. A. Wheeler), Benjamin, New York, 1968, 84.

[2] I. M. Gel'fand & G. E. Shilov, *Generalized functions*, Vol. I, Academic Press, New York, 1964.

[3] S. Helgason, *Solvability questions for invariant differential operators*, Group Theoretical Methods in Physics, Academic Press, New York, 1977.

[4] H. P. Jacobsen & M. Vergne, *Wave and Dirac operators and representations of the conformal group*, J. Functional Analysis **24** (1977) 52–106.

[5] J. L. Kazdan & F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry **10** (1975) 113–134.

[6] Y. Kossman, *Degrés conformes des laplaciens et des opérateurs de Dirac*, C. R. Acad. Sci. Paris **280** (1975) A-283.

[7] P. D. Lax & R. S. Phillips, *An example of Huygen's principle*, Comm. Pure Appl. Math. **31** (1978) 415–423.

[8] C. Moreno, *On quantization of free fields in stationary space-times*, Lett. Math. Phys. **1** (1977) 407–416.

[9] R. Parthasarathy, *Dirac operators and the discrete series*, Ann. Math. **96** (1972) 1–80.

[10] I. E. Segal, *Mathematical cosmology and extragalactic astronomy*, Academic Press, New York, 1976.

[11] B. Speh, *Composition series for degenerate principal series representations of* $SU(2,2)$, preprint, Massachusetts Institute of Technology, 1976.

[12] B. Ørsted, *Conformally invariant differential equations and projective geometry*, J. Functional Analysis.

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